

Spinors for Spinning p -Branes

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Abstract

The group of the p -brane world volume preserving diffeomorphism is considered. The infinite-dimensional spinors of this group are related, by the nonlinear realization techniques, to the corresponding spinors of its linear subgroup, that are constructed explicitly. An algebraic construction of the Virasoro and Neveu-Schwarz-Ramond algebras, based on this infinite-dimensional spinors and tensors, is demonstrated.

1 Introduction

The subject of extended objects was initiated in the particle/field theory framework by the Dirac action for a closed relativistic membrane as the $(2 + 1)$ -dimensional world-volume swept out in spacetime [1]. It evolved and become one of the central topics following the Nambu-Goto action for a closed relativistic string, as the $(1 + 1)$ -dimensional worldsheet area swept out in spacetime [2, 3]. An important step was the Polyakov action for a closed relativistic string, with auxiliary metric [4], that enabled consequent formulations of the Green-Schwarz superstring [5], and the bosonic, and super p -branes with manifest spacetime supersymmetry [6, 7]. In this work, we follow the original path of the Nambu-Goto-like formulation of the bosonic p -brane and address the question of the spinors of the brane world-volume symmetries. For $p = 1$, these spinors are well known, and represent an important ingredient of the spinning string formulation and the Neveu-Schwarz-Ramond infinite algebras [8, 9].

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There is a direct connection between the spinors appearing in the p -brane formulation and the world spinors of the Metric-Affine [10] and Gauge-Affine [11] theories of gravity, in a generic non-Riemannian spacetime of arbitrary torsion and curvature. This is due to a common geometric and group-theoretic structure of both theories.

In this work we study the topological and group-theoretical features of the brane world volume symmetries relevant for the spinor description, we utilize nonlinear realization techniques to relate these spinors to the ones of the group of linear transformations, and construct explicitly the latter ones. Finally, we demonstrate, in the case of spinning string, a group-theoretical derivation of the Virasoro and Neveu-Schwarz-Ramond algebras, based on algebraic properties of the corresponding infinite-dimensional tensorial and spinorial representations.

2 p -Brane world volume symmetries

Consider a bosonic p -brane embedded in a D -dimensional flat Minkowski spacetime $M^{1,D-1}$. The classical Dirac-Nambu-Goto-like action for p -brane is given by the volume of the world volume swept out by the extended object in the course of its evolution from some initial to some final configuration:

$$S = -\frac{1}{\kappa} \int d^{p+1}\xi \sqrt{-\det \partial_i X^m \partial_j X^n \eta_{mn}} , \quad (1)$$

where $i = 0, 1, \dots, p$ labels the coordinates $\xi^i = (\tau, \sigma_1, \sigma_2, \dots)$ of the brane world volume with metric $\gamma_{ij}(\xi)$, and $\gamma = \det(\gamma_{ij})$; $m = 0, 1, \dots, D-1$ labels the target space coordinates $X^m(\xi^i)$ with metric η_{mn} . The world volume metric $\gamma_{ij} = \partial_i X^m \partial_j X^n \eta_{mn}$ is induced from the spacetime metric η_{mn} .

The Poincaré $P(1, D-1)$ group, i.e. its homogeneous Lorentz subgroup $SO(1, D-1)$, are the physically relevant spacetime symmetries, while the $(p+1)$ -dimensional brane world volume is preserved by the homogeneous volume preserving subgroup $SDiff_0(p+1, R)$ of the General Coordinate Transformation (GCT) group $Diff(p+1, R)$.

The $sdiff_0(p+1, R)$ algebra operators, that generate the $SDiff_0(p+1, R)$ group, are given as follows,

$$sdiff_0(p+1, R) = \left\{ L_{(n)k}^{i_1 i_2 \dots i_{n-1}} = \xi^{i_1} \xi^{i_2} \dots \xi^{i_{n-1}} \frac{\partial}{\partial \xi^k} \mid n = 2, 3, \dots \infty \right\}. \quad (2)$$

Preservation of the world volume requires the $L_{(2)}$ operator to be traceless as achieved by subtracting the dilation operator, i.e. $L_{(2)k}^i = \xi^i \frac{\partial}{\partial \xi^k} - \frac{1}{p+1} \delta_k^i \xi^j \frac{\partial}{\partial \xi^j}$. The $L_{(n)}$, $n = 2, 3, \dots, \infty$, operators are labeled by the $SL(p+1, R)$ subgroup representations given by the Young tableaux $[\lambda_1, \lambda_2, \dots, \lambda_p]$ with $\lambda_1 = 2, 3, \dots, \infty$, and $\lambda_2 = \lambda_3 = \dots = \lambda_p = 1$.

The $SDiff_0(p+1, R)$ commutation relations read:

$$\begin{aligned} & [L_{(m)k}^{i_1 i_2 \dots i_{m-1}}, L_{(n)l}^{j_1 j_2 \dots j_{n-1}}] \\ &= \delta_k^{j_1} L_{(m+n-2)l}^{i_1 i_2 \dots i_{m-1} j_2 j_3 \dots j_{n-1}} + \delta_k^{j_2} L_{(m+n-2)l}^{i_1 i_2 \dots i_{m-1} j_1 j_3 \dots j_{n-1}} + \dots + \delta_k^{j_{n-1}} L_{(m+n-2)l}^{i_1 i_2 \dots i_{m-1} j_1 j_2 \dots j_{n-2}} \\ & - \delta_l^{i_1} L_{(m+n-2)k}^{i_2 i_3 \dots i_{m-1} j_1 j_2 \dots j_{n-1}} - \delta_l^{i_2} L_{(m+n-2)k}^{i_1 i_3 \dots i_{m-1} j_1 j_2 \dots j_{n-1}} - \dots - \delta_l^{i_{m-1}} L_{(m+n-2)k}^{i_1 i_2 \dots i_{m-2} j_1 j_2 \dots j_{n-1} m}. \end{aligned} \quad (3)$$

The above symmetry considerations are purely classical. In the quantum case, the corresponding classical symmetry is modified, up to eventual anomalies, in two ways: (i) the classical group is replaced by its universal covering group, and (ii) the group is minimally extended by the $U(1)$ group of phase factors. The corresponding Lie algebra remains unchanged in the first case, while in the second one, it can have additional central charges.

The feasible ways how to extend the Dirac-Nambu-Goto bosonic p -brane action by the fermionic degrees of freedom are determined by the universal covering group $\overline{SDiff}_0(p+1, R)$ of the $SDiff_0(p+1, R)$ group and the form of its spinorial representations. In the following we address at first with the topological issues that define the type of the universal covering of the $SDiff_0(p+1, R)$ group, and subsequently, we face the problem of the $\overline{SDiff}_0(p+1, R)$ group spinorial representations construction.

3 Existence of the double-covering $\overline{SDiff}_0(p, R)$

Let us state first some relevant mathematical results.

Let $g = k + a + n$ be an Iwasawa decomposition of a semisimple Lie algebra g over R . Let G be any connected Lie group with Lie algebra g , and let K, A, N be the analytic subgroups of G with Lie algebras k, a and n respectively. The mapping $(k, a, n) \rightarrow kan$, ($k \in K, a \in A, n \in N$) is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G , and *the groups A and N are simply connected*.

Any semisimple Lie group can be decomposed into the product of the *maximal compact subgroup K* , an *Abelian group A* and a *nilpotent group N* .

As a result of the above statement, only K is not guaranteed to be simply-connected. There exists a universal covering group \overline{K} of K , and thus also a universal covering of G : $\overline{G} \simeq \overline{K} \times A \times N$.

For the group of volume preserving diffeomorphisms, let $Diff(n, R)$ be the group of all homeomorphisms f of R^n such that f and f^{-1} are of class C^1 . Stewart proved the decomposition $Diff(n, R) = GL(n, R) \times E \times R^n$, where the subgroup H is contractible to a point. In our case the relevant decomposition is $SDiff_0(p+1, R) = SL(p+1, R) \times E \times R^{p+1}$. Thus, as $SO(p+1)$ is the compact subgroup of $SL(p+1, R)$, one finds that $SO(p+1)$ is a deformation retract of $SDiff_0(p+1, R)$.

As a result, there exists a universal covering of the Diffeomorphism group $\overline{SDiff_0}(p+1, R) \simeq \overline{SL}(p+1, R) \times H \times R^{p+1}$.

Summing up, we note that both $SL(p+1, R)$ and $SDiff_0(p+1, R)$ have double coverings, defined by $\overline{SO}(p+1) \simeq Spin(p+1)$ the double-coverings of the $SO(p+1)$ maximal compact subgroup.

The universal covering group \overline{G} of a given group G is a group with the same Lie algebra and with a simply-connected group manifold. A finite dimensional covering, $\overline{SL}(p+1, R)$ i.e. $\overline{SDiff_0}(p+1, R)$, exists provided one can embed $\overline{SL}(p+1, R)$ into a group of finite complex matrices that contain $Spin(p+1)$ as subgroup. A scan of the Cartan classical algebras points to the $SL(p+1, C)$ groups as a natural candidate for the $SL(p+1, R)$ groups covering. However, there is no match of the defining dimensionalities of the $SL(p+1, C)$ and $Spin(p+1)$ groups for $p \geq 2$,

$$\dim(SL(p+1, C)) = p+1 < 2^{\lfloor \frac{p}{2} \rfloor} = \dim(Spin(p+1)),$$

except for $p+1 = 8$. In the $p+1 = 8$ case, one finds that the orthogonal subgroup of the $SL(8, R)$ and $SL(8, C)$ groups is $SO(8)$ and not $Spin(8)$. For a detailed account of the $D = 4$ case cf. [12]. Thus, we conclude that there are no finite-dimensional covering groups of the $SL(p+1, R)$, i.e. $\overline{SDiff_0}(p+1, R)$ groups for any $p \geq 2$. An explicit construction of all spinorial, unitary and nonunitary multiplicity-free [13] and unitary non-multiplicity-free [14], $SL(3, R)$ representations shows that they are indeed all defined in infinite-dimensional spaces.

The universal (double) covering groups of the group $\overline{SDiff_0}(p+1, R)$ and its $\overline{SL}(p+1, R)$ subgroup are, for $p \geq 2$, the groups of infinite complex matrices. All their spinorial representations are infinite dimensional. In the reduction of this representations w.r.t. subgroups $Spin(p+1)$, with a trivial

metric tensor δ , or $Spin(1, p)$, with a Minkowski-like metric tensor η , one has representations of unbounded spin values.

4 The deunitarizing automorphism.

The unitarity properties, that ensure correct physical description of the relevant representations of the $\overline{SDiff}_0(p+1, R)$ and $\overline{SL}(p+1, R)$ groups on quantum states and fields, can be achieved by making use of the unitary (irreducible) representations construction of these groups and the so called "deunitarizing" automorphism of the $\overline{SL}(n, R)$ group. This procedure ensures that in the Special Relativity limit (Lorentz invariance) all physical objects have the usual properties (i.g. boosted electron and/or quark retain their Poincaré properties).

The commutation relations of the $\overline{SL}(p+1, R)$ generators

$$Q_{jk} = i\eta_{jl}L_{(2)k}^l, \quad j, k, l = 0, 1, \dots, p, \quad \eta_{jl} = \text{diag}(+1, -1, \dots, -1), \quad (4)$$

are

$$[Q_{ij}, Q_{kl}] = i(\eta_{jk}Q_{il} - \eta_{il}Q_{kj}), \quad (5)$$

The important subalgebras are as follows.

(i) $so(1, p)$: The $M_{ij} = Q_{[ij]}$ operators generate the Lorentz-like subgroup $\overline{SO}(1, p) \simeq Spin(1, p)$ with $J_{mn} = M_{mn}$ (angular momentum) and $K_m = M_{0m}$ (the boosts) $m, n = 1, 2, \dots, p$.

(ii) $so(p+1)$: The $R_{\hat{i}\hat{j}}$ operators, $\hat{i}, \hat{j} = 1, 2, \dots, p+1$, i.e. J_{mn} and $N_m = Q_{\{0m\}}$ operators generate the maximal compact subgroup $\overline{SO}(p+1) \simeq Spin(p+1)$.

(iii) $sl(p)$: The J_{mn} and $T_{mn} = Q_{\{mn\}}$ operators generate the subgroup $\overline{SL}(p, R)$ - an analog of the "little" group of the massive particle states in Poincaré theory.

The $\overline{SL}(p+1, R)$ commutation relations are invariant under the "deunitarizing" automorphism (originally introduced for the $p=3$ case [12],

$$\begin{aligned} J'_{mn} &= J_{mn}, & K'_m &= iN_m, & N'_m &= iK_m, \\ T'_{mn} &= T_{mn}, & T'_{00} &= T_{00} (= Q_{00}), \end{aligned}$$

so that (J_{mn}, iK_m) generate the new compact $\overline{SO}(p+1)'$ and (J_{mn}, iN_m) generate $\overline{SO}(1, p)'$.

The above deunitarizing automorphism generalizes to the arbitrary signature case. Let $\overline{SL}(n, R)$ group act on $R^{r,s}$, $r + s = n$ with metric $\eta = \text{diag}(+1, \dots, +1, -1, \dots, -1)$ having r times $+1$ and s times -1 on the diagonal. The group generators Q_{ij} split accordingly to Q_{ab} , Q_{mn} , Q_{am} , and Q_{ma} , where $a, b = 1, 2 \dots r$, $m, n = 1, 2, \dots s$. The deunitarizing automorphism, that leaves the $sl(n, R)$ algebra invariant, is given as follows,

$$Q'_{ab} = Q_{ab} \ , \quad Q'_{mn} = Q_{mn} \ , \quad Q'_{am} = iQ_{am} \ , \quad Q'_{ma} = -iQ_{ma} \quad (6)$$

The construction of physically relevant representations is achieved through a two step procedure: (1) One constructs, utilizing the appropriate mathematical theorems and methods, the unitary irreducible spinorial, as well as tensorial, representations of the $\overline{SDiff}_0(p+1, R)$ and $\overline{SL}(p+1, R)$ groups in the basis of the maximal compact $Spin(p+1)$ subgroup representations, and (2) One converts these representations, by making use of the deunitarizing automorphism, to representations that are finite and nonunitary for the physical $Spin(1, p)$ subgroup.

5 Nonlinear $\overline{SDiff}_0(p+1, R)$ representations

The GCT group $SDiff_0(p+1, R)$ is an infinite parameter Lie group with the corresponding infinite algebra that acts linearly, e.g. as infinite matrices, on an infinite dimensional vector space. However, its defining representation is given by the group of volume preserving nonlinear transformations of the R^{p+1} spacetime. The $SDiff_0(p+1, R)$ group being nonlinearly realized over its $SL(p+1, R)$ subgroup.

The defining representation of the $\overline{SDiff}_0(p+1, R)$ universal (i.e. double) covering group, as well as of its $\overline{SL}(p+1, R)$ subgroup, is given, as demonstrated above, by the infinite dimensional matrices. In other words, there are no group of finite complex matrices that is isomorphic to $\overline{SDiff}_0(p+1, R)$.

Let us consider now the spinorial representations of the $\overline{SDiff}_0(p+1, R)$ group. There are genuine linear spinorial representations of the $\overline{SDiff}_0(p+1, R)$ group that are infinite dimensional. Moreover, all of its infinitely many Lie algebra generators are likewise represented linearly by infinite matrices. Besides, there are two distinct classes of $\overline{SDiff}_0(p+1, R)$ nonlinear spinorial realizations characterized by:

(i) $\overline{SDiff}_0(p+1, R)$ group is nonlinearly realized over its maximal linear subgroup $\overline{SL}(p+1, R)$; $\overline{SL}(p+1, R)$ and $Spin(1, p)$ are represented linearly,

(ii) both $\overline{SDiff}_0(p+1, R)$ and its $\overline{SL}(p+1, R)$ subgroup are realized nonlinearly over the orthogonal subgroup $Spin(1, p)$.

We recall now a few basic notions from the nonlinear representations theory [15, 16] and set up required notation. Let G be an n_G parameter Lie group, and let H be an n_H parameter subgroup of G . Let \mathcal{M} be a real analytic manifold of dimension d . The mappings R from $g \times \mathcal{M}$ into \mathcal{M} form a representation of G if, for each $g \in G$, $p \in \mathcal{M}$, there is an element $R(g)[p] \in \mathcal{M}$ such that (i) $R : (g, p) \rightarrow R(g)[p]$ is analytic, (ii) $R(e)[p] = p$, for all $p \in \mathcal{M}$, e is the identity in G , and (iii) $R(g_1)R(g_2)[p] = R(g_1g_2)[p]$, for all $g_1, g_2 \in G$, all $p \in \mathcal{M}$.

At each point $p \in \mathcal{M}$, local coordinates can be introduced by mapping an open neighborhood of p into an open neighborhood of R^d . Let q denotes the coordinates of a general point $p \in \mathcal{M}$, and let α be the group parameters of an element $g \in G$ in a neighborhood of e . Then $R(g)[p]$ can be expressed as an analytic function $r(q, \alpha)$ of both q and α , which is in general nonlinear.

An equivalence of two representations is naturally expressed through an independence of the choice of coordinates. Usually, there exists a special point, base point, on \mathcal{M} which must be represented by the origin q_0 in all coordinates. Thus, one defines a concept of local equivalence. Two representations $R_1(g)$ and $R_2(g)$ are locally equivalent if there exists an (in general nonlinear) operator S from $R^n \rightarrow R^n$ such that (i) $S : q \rightarrow S[q]$ is analytic and has an analytic inverse at q_0 , (ii) $S[R_1(g)][q] = R_2(g)S[q]$, for all $g \in G$ in a suitable neighborhood of the identity, and all q in a neighborhood of q_0 , and (iii) $S[q_0] = q_0$. Representation is said linearizable if it is locally equivalent to a linear representation.

Let H be a subgroup of G such that for each $h \in H$, $R(h)[q_0] = q_0$, i.e. let H be the isotropy subgroup of the origin q_0 . Now, it turns out that a restriction $R(h)$, $h \in H$ of the representation $R(g)$ is locally equivalent to a linear representation. In the expansion $R(g)[q]$, $g = h \in H$ in power series $R(h) = D(h)q + O(q^2)$, one finds a linear representation $D(h)$ of H . The change of coordinates defined by $S : q \rightarrow \bar{q} = S[q] = \int_H dh D^{-1}(h) R(h)[q]$, where dh is the right invariant measure on H , establishes a local equivalence between $D(h)$ and the restriction of $R(g)$ to H , i.e. $R(h)[\bar{q}] = D(h)\bar{q}$.

An arbitrary element g in G can be written as $g = ch$, where h belongs to H and c belongs to the left coset space $C = G/H$. Furthermore, an arbitrary point q of the orbit can be written as $q = R(g)[q_0] = R(c)R(h)[q_0] = R(c)[q_0]$. Thus, the elements of the orbit are in one-to-one correspondence with the elements of the coset space G/H . They form a homogeneous space on which

G can be represented.

An action of an arbitrary element g_1 on c is as follows $g_1 c = c_1 h_1 c = c' h'$. The parameters of the group element h' depend both on the group element g_1 and on c , i.e. $h' = h'(c, g_1)$. The transformation $h \rightarrow h'$ is in general nonlinear, and it becomes linear when g_1 is restricted to H .

Let us choose the generators X_a , $a = 1, 2, \dots, n_H$ of H and the remaining generators Y_b , $b = 1, 2, \dots, n_G - n_H$ of G such that they form together a complete set of generators of G that is orthonormal with respect to the Cartan inner product. In some neighborhood of the identity of G , every element $g \in G$ can be decomposed uniquely as follows

$$g = ch = e^{-i\zeta \cdot Y} e^{-i\omega \cdot X}, \quad \zeta \cdot Y = \zeta^b Y_b, \quad \omega \cdot X = \omega^a X_a, \quad \zeta^b, \omega^a \in R. \quad (7)$$

The ζ^b and ω^a parameters form a real n_G -component vector (ζ, ω) . Now, owing to the fact that H leaves the origin q_0 fixed, the orbit \mathcal{N} of q_0 under G separates the G/H cosets defined by $L_\zeta = e^{-i\zeta \cdot Y}$. One has

$$R(g)[q_0] = R(e^{-i\zeta \cdot Y})R(e^{-i\omega \cdot X})[q_0] = e^{-i\zeta \cdot R(Y)}[q_0],$$

and the dimension of the orbit \mathcal{N} is given by the number of ζ^b parameters, i.e. it is equal to $n_G - n_H$. The simplest choice is to represent the orbit elements by L_ζ . We split now the manifold \mathcal{M} into \mathcal{N} and its orthogonal complement \mathcal{V} , which is $d - (n_G - n_H)$ dimensional, i.e. $\mathcal{M} = \mathcal{N} + \mathcal{V}$. Finally, for the coordinates of \mathcal{M} we write $q = (L_\zeta, \psi)$, $L_\zeta \in \mathcal{N}$, $\psi \in \mathcal{V}$. According to the linearization procedure, we can choose the coordinates (L_ζ, ψ) so that H acts linearly, and in particular the coordinates ψ span a space of a linear representation $D(h)$ of H .

Owing to $g_1 c = c' h' = c' h(c, g_1)$, and $c = L_\zeta$, one finds for L_ζ the following transformation law,

$$g : L_\zeta \rightarrow L_{\zeta'} = g L_\zeta h^{-1}(\zeta, g), \quad g \in G, h \in H, \quad (8)$$

while ψ transforms according to

$$g : \psi \rightarrow \psi' = D(h(\zeta, g))\psi = D(L_{\zeta'}^{-1} g L_\zeta)\psi = D(e^{-i\omega(\zeta, g) \cdot X})\psi. \quad (9)$$

When $g = h$,

$$L_{\zeta'} = h L_\zeta h^{-1} = D^{(\zeta)}(h) L_\zeta, \quad h \in H$$

where $D^{(\zeta)}$ is a linear representation of H in the ζ^b space, while

$$\psi' = D(h)\psi = D(e^{-i\omega \cdot X})\psi.$$

For a linear representation $D(g)$, $g \in G$, one has

$$D(L_\zeta) \rightarrow D(L_{\zeta'}) = D(gL_\zeta h^{-1}(\zeta, g)) = D(g)D(L_\zeta)D(h^{-1}(\zeta, g)).$$

Let Ψ be a basis of this linear representation, i.e. $\Psi' = D(g)\Psi$, $g \in G$. By defining

$$\psi = D(L_\zeta^{-1})\Psi, \quad (10)$$

one relates the linear and nonlinear representations, i.e. one project the linear representation into the corresponding nonlinear one. Indeed, one has

$$\psi \rightarrow \psi' = D(h)\psi, \quad h = h(\zeta, g) = L_{\zeta'}^{-1}gL_\zeta \in H \quad (11)$$

Moreover, one can express the basis Ψ of a linear representation $D(g)$ in terms of the corresponding basis ψ of its nonlinear representation $D(h(\zeta, g))$ as follows

$$\Psi = D(L_\zeta)\psi. \quad (12)$$

5.1 Nonlinear representations over $\overline{SL}(p+1, R)$

Let us consider the case where $\overline{SDiff}_0(p+1, R)$ group is nonlinearly realized over its maximal linear subgroup $\overline{SL}(p+1, R)$. This is a natural extension of $SDiff_0(p+1, R)$ being linearly realized over $SL(p+1, R)$.

As stated above, $\overline{SDiff}_0(p+1, R) = \overline{SL}(p+1, R) \times E \times R^{p+1}$, and thus we have now $g \in G = \overline{SDiff}_0(p+1, R)$, $h \in H = \overline{SL}(p+1, R)$, and $c = L_\zeta \in G/H = E \times R^{p+1}$.

Let ψ transforms w.r.t. a spinorial representation of the $\overline{SL}(p+1, R)$ group, i.e.

$$\psi'_A = \left(D_{\overline{SL}(p+1, R)}(h) \right)_A^B \psi_B, \quad h \in \overline{SL}(p+1, R) \quad A, B = 1, 2, \dots \infty \quad (13)$$

where the index that enumerates the components of ψ runs over an infinite range due to the fact that the spinorial representations of the $\overline{SL}(p+1, R)$ group are for $p+1 \geq 3$ necessarily infinite dimensional. The $\overline{SDiff}_0(p+1, R)$ spinor Ψ transforms as follows

$$\Psi'_{\tilde{A}} = \left(D_{\overline{SDiff}_0(p+1, R)}(g) \right)_{\tilde{A}}^{\tilde{B}} \Psi_{\tilde{B}}, \quad g \in \overline{SDiff}_0(p+1, R), \quad \tilde{A}, \tilde{B} = 1, 2, \dots \infty \quad (14)$$

The $D_{\overline{Diff}_0(p+1,R)}$ representations can be reduced to direct sum of infinite dimensional $D_{\overline{SL}(p+1,R)}$ representations. We consider here those representations of $\overline{Diff}_0(D, R)$ that are nonlinearly realized over the maximal linear subgroup $\overline{SL}(D, R)$.

Provided the relevant $D_{\overline{SL}(p+1,R)}$ spinorial representations are known, one can first define the corresponding spinors, ψ_A , and than make use of the infinite-component pseudo-frames

$$E_{\tilde{A}}^A = (D(L_\zeta))_{\tilde{A}}^A \quad (15)$$

to achieve the required linear-to-nonlinear mapping [17]

$$\Psi_{\tilde{A}} = E_{\tilde{A}}^A(x)\Psi_A, \quad E_{\tilde{A}}^A \sim \overline{Diff}_0(p+1, R)/\overline{SL}(p+1, R) \quad (16)$$

The pseudo-frames $E_{\tilde{A}}^A$ infinitesimal transformations are given by

$$\delta_{\overline{SL}(p+1,R)} E_{\tilde{A}}^A = i\epsilon_j^i \{Q_i^j\}_B^A E_{\tilde{A}}^B \quad (17)$$

where ϵ_j^i and Q_i^j are the group parameters and generators of $\overline{SL}(p+1, R)$, respectively.

The above outlined construction allows one to define a $\overline{Diff}(p+1, R)$ covariant Dirac-like wave equation for the corresponding spinor Ψ provided a Dirac-like wave equation for the $\overline{SL}(p+1, R)$ group is known. In other words, one can lift up an $\overline{SL}(p+1, R)$ covariant equation of the form

$$(i(\Gamma_{\overline{SL}(p+1)}^k)_A^B \partial_k - m)\psi_B = 0, \quad k = 0, 1, \dots, p \quad (18)$$

to a $\overline{Diff}(p+1, R)$ covariant equation

$$(iE_{\tilde{A}}^A(\Gamma_{\overline{SL}(p+1)}^k)_A^B E_{\tilde{B}}^{\tilde{B}} \partial_k - m)\Psi_{\tilde{B}} = 0, \quad k = 0, 1, \dots, p \quad (19)$$

where the former equation exists provided a spinorial $\overline{SL}(p+1, R)$ representation for ψ is given, such that the corresponding representation Hilbert space is invariant w.r.t. $\Gamma_{\overline{SL}(p+1)}^i$ action. The crucial step towards a Dirac-like GCT spinor equation is a construction of the vector operator $\Gamma_{\overline{SL}(p+1)}^i$ in the space of $\overline{SL}(p+1, R)$ spinorial representations. We have recently presented an explicite construction of the Diffeomorphism covariant Dirac-like equation in the $p+1=3$ dimensional case [18].

5.2 Nonlinear representations over $Spin(1, p)$

Let us consider now the case where $\overline{SDiff}_0(p+1, R)$ group is nonlinearly realized over its maximal compact subgroup $Spin(p+1)$ or over the related, physically more interesting, Lorentz-like group $Spin(1, p)$.

The relevant group decompositions are: $\overline{SDiff}_0(p+1, R) = \overline{SL}(p+1, R) \times E \times R^{p+1}$, and the Iwasawa decomposition $\overline{SL}(p+1, R) = Spin(1, p) \times A_{p+1} \times N_{p+1}$, where A_{p+1} and N_{p+1} are the groups of $(p+1) \times (p+1)$ Abelian and nilpotent matrices, respectively. Therefore, $g \in G = \overline{SDiff}_0(p+1, R)$, $h \in H = Spin(1, p)$, and $c = L_\zeta \in G/H = E \times R^{p+1} \times A_{p+1} \times N_{p+1}$.

Here, ψ transforms w.r.t. a spinorial representation of the $Spin(1, p)$ group, i.e.

$$\psi'_\alpha = (D_{Spin(1,p)}(h))^\beta_\alpha \psi_\beta, \quad h \in Spin(1, p) \quad \alpha, \beta = 1, 2, \dots, \dim(D_{Spin(1,p)}),$$

where the indices α, β enumerate the finite-dimensional nonunitary or infinite-dimensional unitary $Spin(1, p)$ representation spaces.

The $\overline{SDiff}_0(p+1, R)$ spinor Ψ transforms as in the previous case. The $D_{\overline{Diff}_0(p+1, R)}$ representations can be reduced to a direct sum of finite-dimensional or infinite-dimensional $D_{Spin(1,p)}$ representations.

Owing to the fact that, in this case, both $\overline{SDiff}_0(p+1, R)$ and $\overline{SL}(p+1, R)$ groups are represented nonlinearly over $Spin(1, p)$, one has that both $SDiff_0(p+1, R)$ and $SL(p+1, R)$ groups are represented nonlinearly over $SO(1, p)$ as well. Therefore, in this case there are no usual, linearly transforming, $SL(p+1, R)$ tensorial quantities. Therefore, this case seems to be of no importance for a spinning p -brane formulation because it fails to provide for a group-theoretical formulation of the bosonic theory sector.

6 $\overline{SL}(p+1, R)$ representations construction

We face now the problem of constructing the (unitary) infinite-dimensional spinorial and tensorial representations of the $\overline{SL}(p+1, R)$ group. The $\overline{SL}(p+1, R)$ group can be contracted (a la Wigner-Inönü) w.r.t. its $Spin(p+1)$ subgroup to yield the semidirect-product group $\hat{T} \wedge Spin(p+1)$. \hat{T} is an $\frac{1}{2}(p)(p+3)$ parameter Abelian group generated by operators $U_{ij} = \lim_{\varepsilon \rightarrow 0} (\varepsilon T_{ij})$, which form a $Spin(p+1)$ second rank symmetric operator obeying the following commutation relations,

$$[J_{ij}, J_{kl}] = -i\delta_{ik}J_{jl} + i\delta_{il}J_{jk} + i\delta_{jk}J_{il} - i\delta_{jl}J_{ik},$$

$$\begin{aligned} [J_{\hat{i}\hat{j}}, U_{\hat{k}\hat{l}}] &= -i\delta_{\hat{i}\hat{k}}U_{\hat{j}\hat{l}} - i\delta_{\hat{i}\hat{l}}U_{\hat{j}\hat{k}} + i\delta_{\hat{j}\hat{k}}U_{\hat{i}\hat{l}} + i\delta_{\hat{j}\hat{l}}U_{\hat{i}\hat{k}}, \\ [U_{\hat{i}\hat{j}}, U_{\hat{k}\hat{l}}] &= 0. \end{aligned} \quad (20)$$

An efficient way of constructing explicitly the $\overline{SL}(p+1, R)$ unitary infinite-dimensional representations is given by the so called "decontraction" formula, which is an inverse of the Wigner-Inönü contraction. According to the decontraction formula, the following operators

$$T_{\hat{i}\hat{j}} = rU_{\hat{i}\hat{j}} + \frac{i}{2\sqrt{U \cdot U}} [C_2(Spin(p+1)), U_{\hat{i}\hat{j}}], \quad (21)$$

together with $J_{\hat{i}\hat{j}}$ form the $\overline{SL}(p+1, R)$ algebra. The parameter r is an arbitrary complex number, $r \in C$, and $C_2(Spin(p+1))$ is the $Spin(p+1)$ second-rank Casimir operator.

For the representation Hilbert space we take the homogeneous space of L^2 functions of the maximal compact subgroup $Spin(p+1)$ parameters. The $Spin(p+1)$ representation labels are given either by the Dynkin labels $(\lambda_1, \lambda_2, \dots, \lambda_q)$ or by the highest weight vector which we denote by $\{j\} = \{j_1, j_2, \dots, j_q\}$, $q = \lfloor \frac{p+1}{2} \rfloor$. The $\overline{SL}(p+1, R)$ commutation relations are invariant w.r.t. an automorphism defined by:

$$s(J) = +J, \quad s(T) = -T. \quad (22)$$

This allows us to associate an 's-parity' to each $Spin(p+1)$ representation contained in an $\overline{SL}(p+1, R)$ representation. In terms of the Dynkin labels we find

$$\begin{aligned} s(D_2) &= (-)^{\frac{1}{2}(\lambda_1 + \lambda_2 - \epsilon)}, \\ s(D_{n \geq 3}) &= (-)^{\lambda_1 + \lambda_2 + \dots + \lambda_{n-2} + \frac{1}{2}(\lambda_n - \lambda_{n-1} - \epsilon)} \\ s(B_1) &= (-)^{\frac{1}{2}(\lambda_1 - \epsilon)} \\ s(B_{n \geq 2}) &= (-)^{\lambda_1 + \lambda_2 + \dots + \lambda_{n-1} + \frac{1}{2}(\lambda_n - \epsilon)} \end{aligned} \quad (23)$$

where $\epsilon = 0$ and $\epsilon = 1$ for λ even and odd, respectively, and D and B refer to Cartan's Lie algebra notation.

The s-parity of the $\frac{1}{2}(p)(p+3)$ -dimension representation $(20 \dots 0) = \square\square$ of $Spin(p+1)$ is: $s(\square\square) = +1$. A basis of an $Spin(p+1)$ irreducible representation is provided by the Gel'fand-Zetlin pattern characterized by the maximal weight vectors of the subgroup chain $Spin(p+1) \supset Spin(p) \supset \dots \supset Spin(2)$.

We write the basic vectors as $\left| \begin{smallmatrix} \{j\} \\ \{m\} \end{smallmatrix} \right\rangle$, where $\{j\}$ are the $Spin(p+1)$ group labels, and the additional labels $\{m\}$ corresponds to $Spin(p) \supset Spin(p-1) \supset \dots \supset Spin(2)$ subgroup chain weight vectors.

The Abelian group generators $\{U\} = \{U_{\{\mu\}}^{\{\square\}}\}$, $\{\mu\} = 1, 2, \dots, \frac{1}{2}(p)(p+3)$, can be, in the case of multiplicity free representations, written in terms of the $Spin(p+1)$ -Wigner functions as follows,

$$U_{\{\mu\}}^{\{\square\}} = D_{\{0\}\{\mu\}}^{\{\square\}}(\phi), \quad (24)$$

ϕ being $Spin(p+1)$ group parameters (e.g. Euler angles).

It is now rather straightforward to determine explicitly the non-compact operators matrix elements, which are given by the following expression:

$$\left\langle \begin{smallmatrix} \{j'\} \\ \{m'\} \end{smallmatrix} \left| T_{\{\mu\}}^{\{\square\}} \right| \begin{smallmatrix} \{j\} \\ \{m\} \end{smallmatrix} \right\rangle = \left(\begin{smallmatrix} \{j'\} & \{\square\} & \{j\} \\ \{m'\} & \{\mu\} & \{m\} \end{smallmatrix} \right) \langle \{j'\} || T^{\{\square\}} || \{j\} \rangle, \quad (25)$$

$$\begin{aligned} \langle \{j'\} || T^{\{\square\}} || \{j\} \rangle &= \sqrt{\dim\{j'\}\dim\{j\}} \left\{ r + \frac{1}{2}(C_2(\{j'\}) - C_2(\{j\})) \right\} \\ &\times \left(\begin{smallmatrix} \{j'\} & \{\square\} & \{j\} \\ \{0\} & \{0\} & \{0\} \end{smallmatrix} \right). \end{aligned} \quad (26)$$

$\left(\begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{smallmatrix} \right)$ is the appropriate "3j" symbol for the $Spin(p+1)$ group. The (unitary) infinite-dimensional representations of the $\overline{SL}(p+1, R)$ algebra are given by these expressions of the non-compact generators together with the well known representation expressions for the maximal compact $Spin(p+1)$ algebra generators. Finally, we apply the deunitarizing automorphism for a correct physical interpretation.

The very fact that the $\overline{SL}(p+1, R)$ generators are constructed in the basis of the maximal compact subgroup $Spin(p+1)$, i.e. in the Hilbert space of square integrable functions, guarantees that they can be exponentiated to the corresponding $\overline{SL}(p+1, R)$ group representations,

$$D_{\overline{SL}(p,R)}(e^{-i\zeta^{jk}T_{jk}}e^{-i\omega^{jk}J_{jk}}) = e^{-i\zeta^{jk}D_{\overline{SL}(p,R)}(T_{jk})}e^{-i\omega^{jk}D_{\overline{SL}(p,R)}(J_{jk})}. \quad (27)$$

In the case of the multiplicity free $\overline{SL}(p+1, R)$ representations, each $Spin(p+1)$ sub-representation appears at most once and has the same s -parity. This feature is especially useful for the task of reducing infinite-dimensional spinorial and tensorial representations of the $\overline{SL}(p+1, R)$ group to the corresponding $\overline{SL}(p, R)$ subgroup representations.

We present now just a few examples of the simplest $\overline{SL}(p+1, R)$ spinorial representations in terms of the corresponding $Spin(p+1)$ subgroup representations.

$$\begin{aligned}
p = 2 : \quad D_{\overline{SL}(3,R)} &\supset D_{Spin(3)}^2 \oplus D_{Spin(3)}^6 \oplus D_{Spin(3)}^{10} \oplus \dots, \\
p = 3 : \quad D_{\overline{SL}(4,R)} &\supset D_{Spin(4)}^2 \oplus D_{Spin(4)}^6 \oplus D_{Spin(4)}^{12} \oplus \dots, \\
p = 4 : \quad D_{\overline{SL}(5,R)} &\supset D_{Spin(5)}^4 \oplus D_{Spin(5)}^{40} \oplus D_{Spin(5)}^{140} \oplus \dots, \\
p = 7 : \quad D_{\overline{SL}(8,R)} &\supset D_{Spin(8)}^8 \oplus D_{Spin(8)}^{56} \oplus D_{Spin(8)}^{224} \oplus \dots, \\
p = 9 : \quad D_{\overline{SL}(10,R)} &\supset D_{Spin(10)}^{16} \oplus D_{Spin(10)}^{144} \oplus D_{Spin(10)}^{720} \oplus \dots,
\end{aligned}$$

where the $Spin(p+1)$ representation superscript denotes its dimensionality.

7 The Spinning string case

Let us finally address the question of a group-theoretical approach to construction of spinning p -brane infinite-dimensional Lie algebras that generalize the Virasoro, and Neveu-Schwarz-Ramond algebras, and superalgebras, respectively.

Fradkin and Linetsky [19] proposed a method of constructing infinite-dimensional Lie algebras (of the Virasoro type) by analytic continuation of the finite classical algebras in the space of weight diagrams. This method fails for $\overline{Diff}_0(p+1, R)$ and/or $\overline{SL}(p+1, R)$ algebras, since in these cases there are no finite-dimensional weight diagrams to be continued to an infinite system.

We have explicitly constructed above the infinite-dimensional spinorial and tensorial representations of the $\overline{SL}(p+1, R)$ group, over which the full p -brane invariance $\overline{SDiff}_0(p+1, R)$ is realized nonlinearly. There are two relevant facts: (i) an action of the $\overline{SDiff}_0(p+1, R)$ generators leaves the $\overline{SL}(p+1, R)$ group representation space $\mathcal{V}_{\overline{SL}(p+1,R)}$ invariant, and (ii) the $\overline{SDiff}_0(p+1, R)$ generators $L_{(n)k}^{i_1 i_2 \dots i_{n-1}}$, $n = 2, \dots, \infty$ transform w.r.t. $\overline{SL}(p+1, R)$ subalgebra generators $L_{(2)k}^i$ as components of an irreducible tensor operator.

On the basis of these two facts, we propose the following procedure to construct the infinite p -brane Lie algebras/superalgebras:

- (a) Introduce an infinite set of operators characterized by the $\overline{SL}(p+1, R)$ group representation labels,
- (b) Require these operators to have commutation relations with the $L_{(2)k}^i$ generators as components of an irreducible tensor operator, and
- (c) Demand that these operators satisfy mutually, as well as with the $\overline{SL}(p+1, R)$ generators, the (graded) Jacobi relations.

We demonstrate now this three steps procedure in the well known, $p = 1$, case of the spinning string Virasoro and Neveu-Schwarz-Ramond algebras.

7.1 Irreducible representations of the $\overline{SL}(2, R)$ group

The commutation relations of the $\overline{SL}(2, R)$ algebra $\{J_0, T_{\pm} = T_1 \pm T_2\}$ read

$$[J_0, T_{\pm}] = \pm T_{\pm} \quad [T_+, T_-] = -2J_0.$$

According to the Iwasawa decomposition, $G = NAK$, where N , A , K are nilpotent, Abelian and maximal compact subgroups respectively. Any group element $g \in G$ can be written as

$$g = n(\nu)a(\lambda)k(\gamma) = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \exp(\frac{\lambda}{2}) & 0 \\ 0 & \exp(-\frac{\lambda}{2}) \end{pmatrix} \begin{pmatrix} \cos(\frac{\gamma}{2}) & -\sin(\frac{\gamma}{2}) \\ \sin(\frac{\gamma}{2}) & \cos(\frac{\gamma}{2}) \end{pmatrix}.$$

The differential forms of the group generators and the Casimir operator, in terms of the above parameters, are

$$J_0 = i \frac{\partial}{\partial \gamma}, \quad T_{\pm} = e^{\mp i \gamma} \left(i \frac{\partial}{\partial \lambda} \mp \frac{\partial}{\partial \gamma} \right); \quad C^2 = \frac{\partial}{\partial \lambda} \left(\frac{\partial}{\partial \lambda} - 1 \right).$$

The generators matrix elements, in the J_0 eigenstate basis $f_m(\gamma) = \langle \gamma | m \rangle$, $m = 0, \pm \frac{1}{2}, \dots$ ($\frac{\partial}{\partial \lambda} \rightarrow a$) are as follows:

$$J_0 |m\rangle = m |m\rangle, \quad T_{\pm} |m\rangle = i(a \pm m) |m \pm 1\rangle; \quad C^2 |m\rangle = a(a-1) |m\rangle \quad \forall a.$$

7.2 Infinite bosonic algebra - Virasoro algebra

Let $\{E_m | m = 0, \pm 1, \pm 2 \dots\}$ be an infinite set of operators, such that $[E, E] \subset E$, which transform as components of $\overline{SL}(2, R)$ irreducible tensor operator,

$$[J_0, E_m] = m E_m, \quad [T_{\pm}, E_m] = i(a \pm m) E_{m \pm 1}.$$

The Jacobi relation for (J_0, E_m, E_n) implies

$$[E_m, E_n] = A_{m,n}E_{m+n} + C_m\delta_{m+n,0},$$

while the Jacobi relation for (T_+, E_m, E_n) implies

$$\begin{aligned} (a+m+n)A_{m,n} &= (a+m)A_{m+1,n} + (a+n)A_{m,n+1} \\ (a+m)C_{m+1} + (a+n)C_m &= 0 \quad m+n+1=0. \end{aligned}$$

There is a solution of these relations for $a = -1$, and finally, we arrive at the Virasoro algebra, i.e.

$$[E_m, E_n] = (m-n)E_{m+n} + dm(m^2-1)\delta_{m+n+1,0}, \quad d \in R. \quad (28)$$

7.3 Infinite super algebra - Neveu-Schwarz-Ramond superalgebra

Let $\{E_m | m = 0, \pm 1, \pm 2 \dots\}$, and $\{S_\mu | \mu = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots\}$, be infinite sets of operators, such that $[E, E] \subset E$, $[E, S] \subset S$ and $\{S, S\} \subset E$, which transform as components of $\overline{SL}(2, R)$ irreducible tensor operators,

$$\begin{aligned} [J_0, E_m] &= mE_m, & [T_\pm, E_m] &= i(a \pm m)E_{m\pm 1}, \\ [J_0, S_\mu] &= \mu S_\mu, & [T_\pm, S_\mu] &= i(b \pm \mu)S_{\mu\pm 1} \end{aligned}$$

The Jacoby relation for (J_0, S_μ, S_ν) implies

$$\{S_\mu, S_\nu\} = B_{\mu,\nu}E_{\mu+\nu} + D_\mu\delta_{\mu+\nu,0},$$

while the Jacobi relation for (T_+, S_μ, S_ν) implies

$$\begin{aligned} (a+\mu+\nu)B_{\mu,\nu} &= (b+\mu)B_{\mu+1,\nu} + (b+\nu)B_{\mu,\nu+1} \\ (b+\mu)D_{\mu+1} + (b+\nu)D_\mu &= 0 \quad \mu+\nu+1=0. \end{aligned}$$

There is a solution of these equations for $b = -\frac{1}{2}$,

$$\{S_\mu, S_\nu\} = 2E_{\mu+\nu} + d'(\mu^2 - \frac{1}{4})\delta_{\mu+\nu,0}.$$

The Jacobi relation for (J_0, E_m, S_μ) implies

$$[E_m, S_\mu] = F_{m,\mu}S_{m+\mu},$$

while the Jacobi relation for (T_+, E_m, S_μ) implies

$$(b + m + \mu)F_{m,\mu} = (a + m)F_{m+1,\mu} + (b + \mu)F_{m,\mu+1},$$

and for $a = -1$, $b = -\frac{1}{2}$ one has

$$F_{m,\mu} = \left(\frac{m}{2} - \mu\right) F.$$

The Jacobi relation for (E_m, E_n, S_μ) implies

$$\left(\frac{n}{2} - \mu\right) \left(\frac{m}{2} - n - \mu\right) F^2 = (m-n) \left(\frac{m}{2} + \frac{n}{2} - \mu\right) F + \left(\frac{m}{2} - \mu\right) \left(\frac{n}{2} - m - \mu\right) F^2$$

For $F = 1$ one has

$$[E_m, S_\mu] = \left(\frac{m}{2} - \mu\right) S_{m+\mu}.$$

The Jacobi relation for (S_μ, E_m, S_ν) implies

$$d' \left(\frac{m}{2} - \nu\right) \left(\mu^2 - \frac{1}{4}\right) = d' \left(\mu - \frac{m}{2}\right) \left((m + \mu)^2 - \frac{1}{4}\right) + 2dm(m^2 - 1),$$

i.e. $d' = 4d$

Finally, we obtain the Neveu-Schwarz-Ramond superalgebra:

$$\begin{aligned} [E_m, E_n] &= (m - n)E_{m+n} + dm(m^2 - 1)\delta_{m+n+1,0}, \\ [E_m, S_\mu] &= \left(\frac{m}{2} - \mu\right) S_{m+\mu}, \\ \{S_\mu, S_\nu\} &= 2E_{\mu+\nu} + 4d \left(\mu^2 - \frac{1}{4}\right) \delta_{\mu+\nu,0} \quad d \in R. \end{aligned} \tag{29}$$

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